Effect of Environment on the Elastic Response of Layered Composite Plates

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A generalized Duhamel-Neumann form of Hooke's law is used to develop laminated plate equations which include the effect of expansional strains. Such strains are induced in composite materials by temperature, absorption by a polymeric matrix material of a swelling agent such as water vapor, and by sudden expansion of absorbed gases in the matrix. Solutions to specific boundary value problems are presented for both symmetric and nonsymmetric laminates. Numerical results indicate that in addition to inducing residual stresses, expansional strains can substantially affect the gross response characteristics of a composite material.

Introduction

HIGH-PERFORMANCE composite materials have received increasing consideration for structural applications because of their low density, high strength, and high stiffness. Their superior strength and stiffness properties, however, are often compromised by the environment to which they are exposed. A general discussion of the affect of environment on the structural behavior of composite materials has been presented by Halpin.[1] Among the environmental factors, those which induce expansional strains (volume change in the absence of surface tractions) are of particular concern. In the case of advanced composite structures, such phenomena are primarily caused by an increase in temperature, absorption by a polymeric matrix material of a swelling agent such as water vapor, and by sudden expansion of absorbed gases in the matrix. In addition to inducing residual stresses, expansional strains can effect the gross response characteristics of a composite structure. In particular, bending deflections, buckling loads, and vibration frequencies can be considerably modified by the presence of environmentally induced strains. Thus, if composite materials are to reach their full potential, it will be necessary to consider such environmental factors as temperature and humidity in structural analysis and design.

Although thermal effects have been included in the formulation of anisotropic laminated plate theory,[2,3] solutions to specific boundary value problems have been limited to the bending of an infinite strip. Furthermore, current lamination theory does not provide for the general case of expansional strains. Halpin and Pagano[4] presented a generalized Duhamel-Neumann form of Hooke's law applicable to a wide variety of environmental problems. This approach was used in conjunction with concepts of physical chemistry, micromechanics, and laminated plate theory to predict the swelling strains in a layered composite of symmetric construction. Analytical results were experimentally verified.

It is the purpose of the present paper to extend the work of Halpin and Pagano[4] to the general case of laminated plates and to present solutions to specific boundary value problems. Analytical results are used to show the effect of expansional strains on the bending, buckling, and vibrations of current high-performance composite laminates.

Received March 29, 1971; presented as Paper 71-352 at the AIAA/ASME 12th Structures, Structural Dynamics and Materials Conference, Anaheim, Calif., April 19–21, 1971; revision received May 26, 1971. The authors wish to acknowledge M. Younger of the Aeronautical Systems Division, Wright-Patterson Air Force Base for her assistance in the computer programming.

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Index Category: Structural Composite Materials (Including Coatings); Thermal Stresses.

Governing Equations

Consider a thin plate of constant thickness h composed of layers of anisotropic materials bonded together. The plate is referred to a standard cartesian coordinate system located in the midplane (x-y) with the Z-axis normal to this plane. The material in each layer is assumed to have a plane of elastic symmetry parallel to x-y.

In contracted notation the generalized Duhamel-Neumann form of Hooke's law for plane stress is[4]

$$\epsilon_i = S_{ij}\sigma_j + \bar{\epsilon}_i \qquad (i,j = 1,2,6) \tag{1}$$

where the stresses are denoted by $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_6 = \sigma_{xy}$, and the engineering strains ϵ_i are defined in an analogous manner. Analogous expansional strains are denoted by $\bar{\epsilon}_i$, i.e.,

$$\bar{\epsilon}_i = \bar{\epsilon}_{i(\text{thermal})} + \bar{\epsilon}_{i(\text{swelling})} + \ldots \tag{2}$$

The inverted form of Eq. (1) is

$$\sigma_i = Q_{ij}(\epsilon_i - \bar{\epsilon}_j) \qquad (i,j = 1,2,6) \tag{3}$$

where Q_{ij} are elements of the anisotropic reduced stiffness matrix for plane stress.

As a consequence of the Kirchhoff-Love hypothesis[3,5,6]

$$\epsilon_i = \epsilon_i{}^0(x,y,t) + z\kappa_i(x,y,t) \qquad (i = 1,2,6,) \tag{4}$$

where t denotes time, $\epsilon_i{}^0$ are the midplane strains, and κ_i are the plate curvatures. Plate force and moment resultants are defined in the usual manner, i.e.,

$$(N_i, M_i) = \int_{-h/2}^{h/2} \sigma_i(1,z)dz \qquad (i = 1,2,6) \tag{5}$$

where N_i and M_i are force and moment resultants, respectively. Substitution of Eq. (3) into Eq. (5) and taking Eq. (4)

into account, leads to the plate constitutive relations

$$N_i = A_{ij}\epsilon_j^0 + B_{ij}\kappa_j - \bar{N}_i \quad (i = 1,2,6) \quad (6)$$
$$M_i = B_{ij}\epsilon_j^0 + D_{ij}\kappa_j - \bar{M}_i$$

where

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1,z,z^2)dz$$
$$(\bar{N}_i,\bar{M}_i) = \int_{-h/2}^{h/2} Q_{ij}\bar{\epsilon}_j(1,z)dz$$

It should be noted that for the general case $\bar{\epsilon}_i = \bar{\epsilon}_i(x,y,z,t,T)$, where T is temperature, and is determined from principles of heat transfer in the case of thermal expansion and from consideration of physical chemistry concepts in the case of swelling. It is interesting to note that a necessary condition for the existence of an exact solution to the three-dimensional thermoelastic field equations which satisfies the plane-stress hypothesis for an isotropic material is that the temperature distribution satisfy the equation[7]

$$\nabla^2 T = f(z,t) \quad (7)$$

The derivation of an analogous expression for anisotropic materials is not, however, straightforward and as result would make an interesting subject for a separate study.

From laminated plate theory[3,5]

$$[\epsilon^0] = \begin{bmatrix} u,_x{}^0 \\ v,_y{}^0 \\ u,_y{}^0 + v,_x{}^0 \end{bmatrix}, \quad [\kappa] = \begin{bmatrix} -w,_{xx} \\ -w,_{yy} \\ -2w,_{xy} \end{bmatrix} \quad (8)$$

where u^0 and v^0 are midplane displacements in the x and y direction, respectively, w is the transverse deflection, and a comma denotes partial differentiation. Equation (6) in conjunction with Eq. (8) and the equations of motion of laminated plate theory[3] (with rotary inertia neglected and initial in-plane force effects included) leads to the following displacement equations:

$$L_1u^0 + 2A_{16}u,_{xy}{}^0 + L_2v^0 - L_4w,_x - L_5w,_y - \bar{N}_{x,x} - \bar{N}_{xy,y} = Pu,_{tt}{}^0 \quad (9a)$$

$$L_2u^0 + L_3v^0 + 2A_{26}v,_{xy}{}^0 - L_6w,_x - L_7w,_y - \bar{N}_{xy,x} - \bar{N}_{y,y} = Pv,_{tt}{}^0 \quad (9b)$$

$$-L_4u,_x{}^0 - L_5u,_y{}^0 - L_6v,_x{}^0 - L_7v,_y{}^0 + L_8w,_{xx} + 4D_{16}w,_{xxxy} + 4D_{26}w,_{xyyy} + D_{22}w,_{yyyy} + \bar{M}_{x,xx} + 2\bar{M}_{xy,xy} + \bar{M}_{y,yy} + Pw,_{tt} = q + N_x{}^iw,_{xx} + 2N_{xy}{}^iw,_{xy} + N_y{}^iw,_{yy} \quad (9c)$$

where P is the integral of mass density through the plate thickness, q is a distributed pressure over the surface of the plate, $N_x{}^i$, $N_y{}^i$ and $N_{xy}{}^i$ are prebuckling stress resultants, and the operators L_i are defined as follows:

$$L_1 = A_{11}(\ ),_{xx} + A_{66}(\ ),_{yy}$$
$$L_2 = A_{16}(\ ),_{xx} + (A_{12} + A_{66})(\ ),_{xy} + A_{26}(\ ),_{yy}$$
$$L_3 = A_{66}(\ ),_{xx} + A_{22}(\ ),_{yy}$$
$$L_4 = B_{11}(\ ),_{xx} + (B_{12} + 2B_{66})(\ ),_{yy}$$
$$L_5 = 3B_{16}(\ ),_{xx} + B_{26}(\ ),_{yy}$$
$$L_6 = B_{16}(\ ),_{xx} + 3B_{26}(\ ),_{yy}$$
$$L_7 = (B_{12} + B_{66})(\ ),_{xx} + B_{22}(\ ),_{yy}$$
$$L_8 = D_{11}(\ ),_{xx} + 2(D_{12} + 2D_{66})(\ ),_{yy}$$

It should be noted that Eqs. (9) are essentially the linear equations of Ref. 3 with expansional resultants $\bar{N}_i$ and $\bar{M}_i$ replacing the thermal resultants $N_i{}^T$ and $M_i{}^T$. Thus, the concept of expansional strain can be easily fit into the framework of laminated plate theory by generalizing the definitions of thermal force and moment resultants.

Analysis of Symmetric Laminates

For laminates having elastic properties which are symmetric with respect to the midplane, all elements of the B_{ij} coupling stiffness matrix vanish, causing the in-plane problem to become uncoupled from the bending problem. In the case of uniform expansional strains within each layer, constant values of $\bar{N}_i$ are obtained from the solution of the in-plane problem. Thus for the purpose of illustrating the effects of expansional strains on the behavior of laminated composites, consideration is limited to a uniform temperature increase in the case of thermal strains and to long time steady-state environmental exposure in the case of swelling.

With the B_{ij} elements all zero, the analysis of symmetrical laminated plates is reduced to the analysis of an anisotropic plate with uniform in-plane loads $\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$ (these in-plane loads are of course proportional to the dilatational strain). Stability, vibration, and bending behavior of such a plate may be conveniently analyzed using an energy formulation in conjunction with the Ritz method. The development of this approach presented below is essentially that given in Ref. 8.

The appropriate energy conditions for the problems at hand are[9]

$$V + U + Q = \text{stationary value} \quad (10)$$

for bending or stability problems, or

$$V + U + Q - T = \text{stationary value} \quad (11)$$

for vibration, where V = potential energy or strain energy of bending the plate; U = potential energy of the inplane loads moving through the bending deflections; Q = potential energy of the transverse loading; T = kinetic energy.

For a symmetric laminated plate with constitutive equations governing bending of

$$M_i = D_{ij}\kappa_j$$

with uniform transverse load q, expansional-strain-induced in-plane loads $\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$, and density ρ, the necessary energy expressions can be written[9]

$$V = \tfrac{1}{2}\iint_{\text{Area}}\{D_{11}(w,_{xx})^2 + 2D_{12}w,_{xx}w,_{yy} + D_{22}(w,_{yy})^2 + 4D_{16}w,_{xy}w,_{xx} + 4D_{26}w,_{xy}w,_{yy} + 4D_{66}(w,_{xy})^2\}d\,\text{Area}$$

$$U = \tfrac{1}{2}\iint_{\text{Area}}\{\bar{N}_x(w,_x)^2 + \bar{N}_y(w,_y)^2 + 2\bar{N}_{xy}w,_xw,_y\}d\,\text{Area}$$

$$Q = \iint_{\text{Area}} qw\ d\,\text{Area}$$

$$T = \omega^2\rho h \iint_{\text{Area}} w^2\ d\,\text{Area}$$

where for vibration only free harmonic vibration at frequency ω is considered.

Since the energy expressions are all given in terms of the displacement w, the energy conditions (10) and (11) are of the form

$$\pi(w) = \text{stationary value} \quad (12)$$

Furthermore, in the present paper, only rectangular plates and hence rectangular integration regions will be considered, so that it is convenient to apply the Ritz method to the energy conditions (10) and (11). To do this, a series with undetermined coefficients for the deflection w is introduced. In the present case, the following form is selected:

$$w = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}X_m(x)Y_n(y) \quad (13)$$

where $X_m(x)$ and $Y_n(y)$ are the characteristic modes of free vibration of beams with appropriate boundary conditions. Introducing the series (13) into expressions (10) or (11), or more generally Eq. (12), we reduce those variational conditions to the following $M \cdot N$ conditions in the discrete variables a_{mn}:

$$\partial\pi(w)/\partial a_{mn} = 0 \qquad (m = 1\ldots M: n = 1\ldots N) \quad (14)$$

In the case of bending due to a uniform lateral load q, Eqs. (14) reduce to a system of linear simultaneous equations for the coefficients a_{mn}. Solution of this set of simultaneous equations gives the coefficients, and these coefficients together with the functions $X_m(x)$ and $Y_n(y)$ provide an equation for the deflection as a function of x and y. In the case of a stability or vibration analysis, the value of the in-plane loads or the vibration frequency is sought such that the system of linear simultaneous equations does not have a unique solution. The lowest value of the in-plane loads that creates this condition is the critical buckling load (in this case the critical buckling strain), while those values of the vibration frequencies that create this condition are the natural vibration frequencies. Of course, the actual evaluation of the buckling load or vibration frequencies can be accomplished with any of the well-known methods related to eigenvalue problems.

The manipulations involved in substituting Eq. (13) into Eqs. (10) or (11), carrying out the differentiations indicated in Eq. (14), and evaluating the necessary integrals that arise, are somewhat tedious, but these operations have been organized such that the task is strictly mechanical, as described in Ref. 8. By appropriate selection of the beam modes $X_m(x)$ and $Y_n(y)$, any edge of a rectangular plate can be considered clamped, simply-supported, or free, again following Ref. 8. Still further expansion on this approach, as well as examples of the convergence for increasing M and N, are given in Refs. 9 and 10.

Analysis of Unsymmetrical Angle-Ply Laminates

Unsymmetrical angle-ply laminates (lay-up containing plies of constant thickness in which the principal elastic axis of symmetry is alternately oriented at $\pm\theta$ to the x axis of the plate) are now considered. For such a laminate $A_{16} = A_{26} = D_{16} = D_{26} = 0$, and B_{16} and B_{26} are the only nonvanishing elements of the coupling stiffness matrix B_{ij}. Again the discussion is limited to a uniform temperature increase in the case of thermal strains and to long time steady-state environmental exposure in the case of swelling. Thus, for uniform expansional strains within each layer $\bar{N}_x$, $\bar{N}_y$, and $\bar{M}_{xy}$ are constant and $\bar{N}_{xy} = \bar{M}_x = \bar{M}_y = 0$. If no external in-plane forces are applied to the plate

$$N_x^i = -\bar{N}_x, \quad N_y^i = -\bar{N}_y, \; N_{xy}^i = 0 \qquad (15)$$

and Eqs. (8) reduce to

$$L_1u^0 + (A_{12} + A_{66})v,_{xy}^0 - L_5w,_y = Pu,_{tt}^0 \qquad (16)$$

$$(A_{12} + A_{66})u,_{xy}^0 + L_3v^0 - L_6w,_x = Pv,_{tt}^0 -$$
$$L_5u,_y^0 - L_6v,_x^0 + L_8w,_{xx} + D_{22}w,_{yyyy} +$$
$$Pw,_{tt} = q - \bar{N}_xw,_{xx} - \bar{N}_yw,_{yy}$$

It should be noted that any expansional strain distribution which is independent of x and y and an even function of z will produce the preceding expansional resultants as well as the ones discussed in the previous section on symmetric laminates.

For a rectangular plate of dimensions a, b subjected to static loading, solutions to Eqs. (14) can be obtained in the form

$$u^0 = a_{mm}\sin(m\pi x/a)\cos(n\pi y/b)$$
$$v^0 = b_{mm}\cos(m\pi x/a)\sin(n\pi y/b) \qquad (17)$$
$$w = c_{mn}\sin(m\pi x/a)\sin(n\pi y/b)$$

Equations (17) satisfy the following simply-supported boundary conditions:

$$\text{at } x = 0 \text{ and } a: \qquad u^0 = N_{xy} = w = M_x = 0 \quad (18)$$
$$\text{at } y = 0 \text{ and } b: \qquad v^0 = N_{xy} = w = M_y = 0 \quad (19)$$

It is obvious that expansion of the functions in Eqs. (17) into a double Fourier series allows Eqs. (16) to be satisfied for any transverse load q which can be represented in the form

$$q(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (20)$$

The coefficients a_{mn}, b_{mn}, and c_{mn} are determined by substituting Eq. (20) and the double series form of Eqs. (17) into Eqs. (16) and solving the resulting simultaneous algebraic equations. Minimum values of $\bar{N}_x$ and $\bar{N}_y$ which cause the coefficient matrix to be singular obviously correspond to expansional strains which induce laminate buckling.

Solutions for the free vibration of unsymmetrical angle-ply laminates subjected to the boundary conditions of Eqs. (18) and (19) are of the form

$$u^0 = [U(x,y)]e^{i\omega t}, \quad v^0 = [V(x,y)]e^{i\omega t}$$
$$w = [W(x,y)]e^{i\omega t} \qquad (21)$$

where U, V, and W are of the same form as Eqs. (17), respectively. Substitution of Eq. (21) into Eqs. (16), with $q = 0$, leads to a standard eigenvalue problem for the determination of the frequency ω. Buckling loads correspond to values of $\bar{N}_x$ and $\bar{N}_y$ for which ω vanishes.

Numerical Examples

The following three sets of fiber-matrix properties are considered:

$$E_{fL}/E_m = 80,\; E_{fT}/E_m = 4,$$
$$G_{fL}/G_m = 22,\; \nu_{fL} = 0.2,\; \nu_m = 0.35 \quad (22a)$$
$$E_f/E_m = 120,\; \nu_f = 0.2,\; \nu_m = 0.35,\; G_f/G_m = 135 \quad (22b)$$
$$E_f/E_m = 25,\; \nu_f = 0.2,\; \nu_m = 0.35,\; G_f/G_m = 28 \quad (22c)$$

where E_f, G_f, ν_f denote the Young's modulus, shear modulus and Poisson's ratio, respectively, of the fiber, and E_m, G_m, and ν_m denote corresponding properties for the matrix.

The first set of data is for transversely isotropic filaments where E_{fL}, E_{fT}, G_{fL}, and ν_{fL} are Young's modulus parallel to the fiber axis, Young's modulus in the radial direction of the fiber, longitudinal shear modulus of the fiber, and the major Poisson's ratio of the fiber, respectively. The data in set one roughly corresponds to high modulus graphite-epoxy, while the second and third sets roughly correspond to boron-epoxy and S glass-epoxy, respectively. Ply properties are calculated from the Halpin-Tsai micromechanics equations.[11] These equations are empirical relationships based on exact elasticity solutions and are reasonably accurate for fiber volume contents of 60% or less. All results presented in this paper are based on 60% fiber volume content in each ply. Off-axis properties are calculated from standard transformation equations.

Thermal Buckling

For the data in Eq. (22) the following thermal expansional properties are assumed:

$$\sigma_{fL}/\sigma_m = -22 \times 10^{-3},\; \alpha_{fT}/\alpha_m = 372 \times 10^{-3} \quad (23a)$$
$$\alpha_f/\alpha_m = 112 \times 10^{-3} \quad (23b)$$
$$\alpha_f/\alpha_m = 64 \times 10^{-3} \quad (23c)$$

where α denotes the linear coefficient of thermal expansion with the subscripts having the same connotation as in Eqs.

(22). The thermal expansion data for graphite fibers is based on work reported by Doner and Novak.[12] The other thermal expansion data can be found in Ref. 13. Ply thermal expansion coefficients are calculated from the approximate micromechanical relations derived by Schapery.[14] Halpin and Pagano[4] showed that Schapery's equations are also valid for swelling strains. For off-axis orientations the thermal coefficients are calculated from the strain transformation equations.

It has been previously shown by Halpin and Pagano[4] that certain symmetric angle-ply orientations of highly anisotropic materials produce a negative expansional strain relative to the x axis (see Fig. 1). In particular, high values of E_L/E_T in conjunction with lamination theory can produce negative effective expansional coefficients for a range of angle-ply orientations. For example, Ref. 4 shows an experimentally measured decrease of 10% in a $\pm 15°$ symmetric angle-ply fiber reinforced rubber laminate subjected to a swelling agent. For this material $E_L/E_T = 132$. The exact value of E_L/E_T above which negative expansions can be demonstrated will depend on $\bar{\epsilon}_f/\bar{\epsilon}_m$. Such a phenomenon has technological significance as it implies that composite laminates can be designed which have a zero or very low expansional coefficient.

As shown in Ref. 4, the effective thermal expansion coefficients for boron-epoxy and glass-epoxy symmetric angle-ply laminates are all positive. Because of the negative thermal expansion coefficient of graphite fibers in the axial direction, unidirectional graphite composites display a negative effective coefficient of thermal expansion relative to the fiber direction. This fact in conjunction with a relatively high value of E_L/E_T yields a range of orientations for which graphite-epoxy symmetric angle-ply laminates have a negative coefficient of thermal expansion. From a practical standpoint, this means that certain orientations of graphite-epoxy angle-ply plates having inplane boundary constraints can be buckled by lowering the temperature rather than raising it. This phenomenon is illustrated in Fig. 1 where nondimensional buckling temperatures are shown for various angle-ply orientations designated as θ along the abscissa. The laminates are 4 layer symmetric (stacking sequence of $+\theta$, $-\theta$, $-\theta$, $+\theta$) and are subjected to a uniform temperature change. The edges at $x = 0$, and $x = a$, are clamped and the adjacent edges at $y = 0$ and $y = b$ are free. For orientations in which $0° \leq \theta < 17.5°$, the laminate has a negative coefficient of thermal expansion relative to the x axis. Thus, in this region the temperature must be decreased in order to induce thermal buckling. At an orientation of $\theta = 17.5°$ the effective thermal expansion coefficient in the x direction vanishes and the plate cannot buckle by either raising or lowering the temperature. For laminates with $17.5° < \theta \leq 90°$, the thermal expansion coefficient in the x direction is positive and the laminate can buckle only by raising the temperature. It should be pointed out, however, that for most practical values of α_m and b/h, the critical buckling temperature is far too low to be of any significance. Thus, for orientations of $0° \leq \theta \leq 17.5°$ it is virtually impossible to buckle the plate by either raising or lowering the temperature.

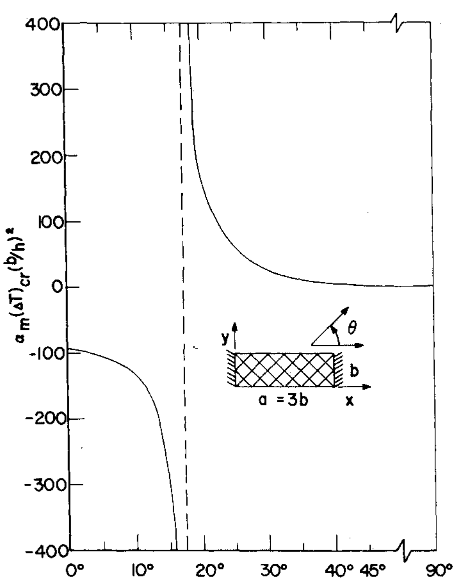

Fig. 1 Thermal buckling of 4 layer symmetric graphite-epoxy angle-ply laminates.

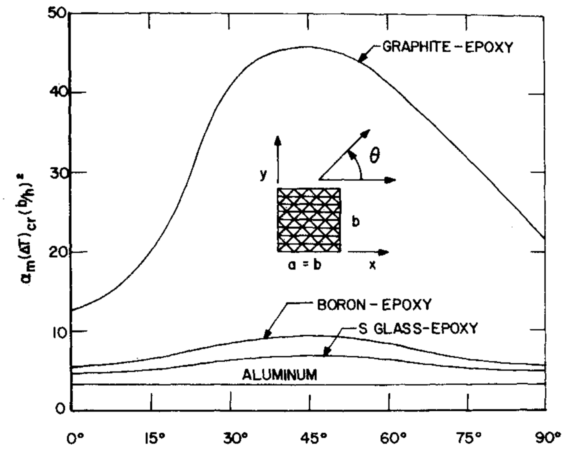

Fig. 2 Thermal buckling of 6 layer symmetric angle-ply laminates.

A cursory examination of Fig. 1 reveals that the thermal expansion coefficient relative to the y axis is positive when the expansion relative to the x axis is negative and vice versa. Thus if such laminates were constrained along all four edges, a temperature change would simultaneously induce both tension and compression in certain orientations. This combination would increase the thermal stability compared to laminates having positive thermal expansion coefficients relative to both the x and y axes. This conclusion is verified analytically in Fig. 2 where a nondimensional buckling temperature is shown for 6 layer symmetric angle-ply laminates ($+\theta$, $-\theta$, $0°$, $0°$, $-\theta$, $+\theta$) under a uniform temperature increase. The various angle-ply orientations θ are plotted along the abscissa, and the plate is simply-supported along all 4 edges. As anticipated, because of the negative value of α_{fL}/α_m, the graphite-epoxy composites have increased stability compared to boron-epoxy and glass-epoxy laminates which display positive effective thermal expansion coefficients relative to the x and y axes of the plate for all angle-ply orientations. Because of the low coefficient of expansion of all the fibers considered, the fiber reinforced materials show increased stability compared to isotropic aluminum for the plates under consideration. Analytical results show that the graphite-epoxy composites display a negative expansion coefficient in the x direction for laminates with $0° \leq \theta < 20°$. As a result, a higher buckling mode corresponding to a decrease in temperature exists in this region. As in the previous case (Fig. 1), however, these critical temperatures are of little practical interest and are not shown in Fig. 2.

Effect of Swelling

Consider composite laminates which are exposed to an increase in relative humidity for a long enough period for equilibrium to be obtained between the material and the environment. A polymeric matrix will absorb moisture and swell. The amount of swelling depends on the degree of molecular cross-linking in the polymer. In general, the fiber-matrix bond is subject to attack in a moisture environment, resulting in a decrease in composite mechanical properties. Consideration in the present paper, however, is limited to the phenom-

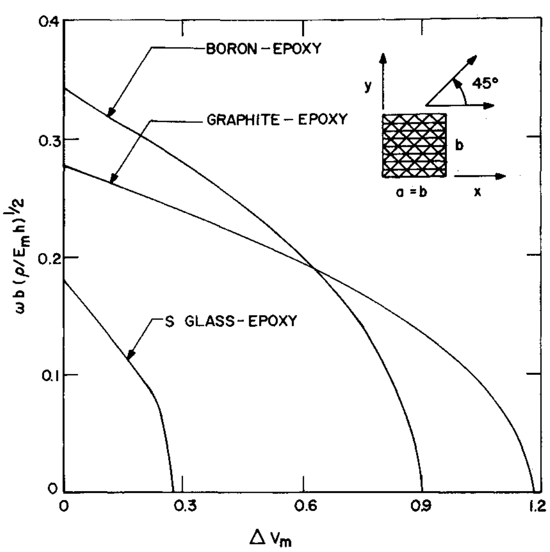

Fig. 3 Effect of matrix swelling on fundamental vibration frequency of 6 layer symmetric angle-ply laminates, a/h = 100.

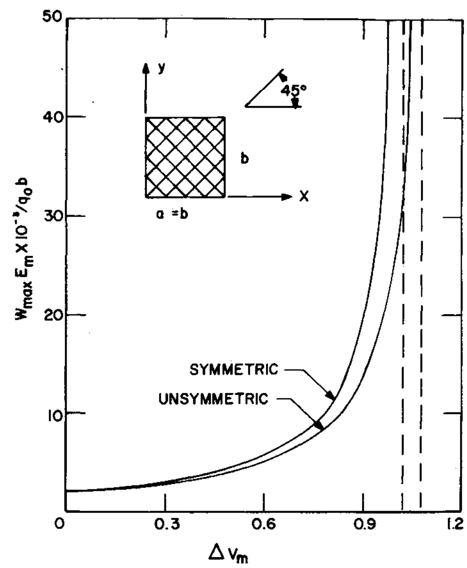

Fig. 4 Effect of matrix swelling on the maximum deflection of uniformly loaded ($q = q_0$) graphite-epoxy laminates a/h = 100.

enon of swelling. As a result, it is assumed that the matrix alone is affected by the increase in humidity.

Matrix swelling produces a gross effect which is directly analogous to thermal expansion. Thus inplane loads are induced for plates with edge constraints. As a result, matrix swelling reduces fundamental vibration frequencies and increases deflections in plate subjected to lateral loads. This is illustrated in both Figs. 3 and 4.

In Figure 3, fundamental vibration frequencies ω are shown in nondimensional form as a function of percent volume change in the matrix ΔV_m due to swelling, where

$$\Delta V_m = [(1 + \bar{\epsilon}_m)^3 - 1](100) \tag{24}$$

and $\bar{\epsilon}_m$ is the matrix expansional strain. The stacking sequence for these laminates is +45°, −45°, 0°, 0°, −45°, +45°. The edges are all simply supported, and the aspect ratio a/h = 100. In each of the laminates considered, an increase in matrix swelling reduces the fundamental vibration frequency. This is due to the fact that matrix swelling leads to positive expansional strains relative to the x and y axes for the laminates under consideration. As a result only compressive inplane loads are induced. The value of ΔV_m at which ω vanishes is the critical buckling strain. To illustrate the importance of swelling, consider that epoxy resins, although highly cross-linked, can undergo as much as a 2% volume change when immersed in water at room temperature for a long period of time. Such a volume change is far more than sufficient magnitude to buckle all of the laminates in Fig. 3. Thus, it is obvious that long time exposure to an environment of moderate change in humidity can significantly effect laminate response. It should be pointed out that other resin systems, such as polyesters, which are not highly cross-linked are also used with glass fibers. For such cases protection in a moisture environment becomes extremely important if phenomena such as "moisture buckling" is to be avoided.

The effect of matrix swelling on the bending deflection of symmetric and unsymmetric 4 layer ±45° graphite-epoxy laminates with a/b = 100 and subjected to a uniform transverse load is illustrated in Fig. 4. Nondimensional maximum deflections are presented as a function of percent volume change of the matrix. The plate is simply supported along all 4 edges and as a consequence matrix swelling induces inplane compressive loads. This results in an increase in bending deflections with an increase in ΔV_m up to the critical buckling value at which point the deflection becomes unbounded. The symmetric laminate stacking sequence is +45°, −45°, −45°, +45°, and for the unsymmetric laminate +45°, −45°, +45, −45°. A cursory examination of Fig. 4 reveals that matrix swelling must be less than 1.25% if buckling is to be prevented. Thus, as in the case of the fundamental vibration frequencies, resin expansion in a humid environment can have serious affects on the bending of composite laminates.

Conclusions

A procedure has been outlined to analyze the effect of environmentally induced strains on the elastic response of layered plates. A generalization of the Duhamel-Neumann form of Hooke's law was used in conjunction with classical laminated plate theory. Results showed that swelling strains can be accounted for in essentially the same manner as thermal strains.

Solutions have been obtained to specific boundary value problems pertaining to the bending, vibrations, and buckling of symmetric and unsymmetric laminates subjected to expansional strains. Results showed that swelling can have considerable effect on the mechanical response of composite materials. Thus, emphasis should be put on the effect of environment in the analysis and design of structures utilizing laminated composites.

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SEPTEMBER 1971 AIAA JOURNAL VOL. 9, NO. 9

Imperfection Sensitivity of Axially Compressed Stringer Reinforced Cylindrical Panels under Internal Pressure

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This paper is a study of the effects of longitudinal edge stiffness and internal pressure on the buckling and initial postbuckling behavior of axially compressed long cylindrical panels. Two methods of solution are presented. The first is a series solution where the torsional rigidity of the edge stiffeners is not included and the second is a numerical solution where the torsional rigidity of the edge stringers is included. The results show that both internal pressure and the edge stiffener torsional rigidity tend to increase the panel buckling strength and the panel insensitivity to initial imperfections.

Nomenclature

A_n,B_n	= coefficients defined in Eq. (35)
a_n,b_n	= series coefficients used in Eqs. (29) and (30)
a,b	= initial postbuckling coefficient defined in Eq. (8)
c	= $\{3(1-\nu^2)\}^{1/2}$
c_0	= arbitrary constant
D	= $Et^3/4c^2$
E	= Young's modulus
e	= constant evolving from Stokes' transformation, see Eq. (31)
$\bar{F}$	= stress function
F	= $2c\bar{F}/(Et^3)$, nondimensional stress function
$F^{(0)},F^{(1)},F^{(2)}$, F_σ,F_p	= defined in Eqs. (3) and (5)
$f(y),f_\alpha(y)$, $f_\beta(y)$	= defined in Eqs. (44) and (19), respectively
K	= initial postbuckling parameter defined in Eq. (13)
G	= shear modulus of the stiffener
J	= torsional constant for the stiffener
$\bar{M}_x,\bar{M}_y,\bar{M}_{xy}$	= bending stress resultants
M_x,M_y,M_{xy}	= $(R/2cEt^3)(\bar{M}_x,\bar{M}_y,\bar{M}_{xy})$, nondimensional bending stress resultants
$\bar{N}_x,\bar{N}_y,\bar{N}_{xy}$	= membrane stress resultant
N_x,N_y,N_{xy}	= $R/(Et^3)\cdot(\bar{N}_x,\bar{N}_y,\bar{N}_{xy})$, nondimensional membrane stress resultants
$\bar{p}$	= applied internal pressure
p	= $\bar{p}R^2c/(Et^2)$
q_0	= $(2cR/t)^{1/2}$
R	= cylindrical panel radius
S	= series summation defined in Eq. (40)
t	= shell thickness
$\bar{U},\bar{V},\bar{W}$	= displacements defined in Fig. 1
U,V	= $q_0/t(\bar{U},\bar{V})$, nondimensional displacements
W	= $\bar{W}/t$, nondimensional displacement
$W^{(0)},W^{(1)},W^{(2)}$, W_σ,W_p	= defined in Eqs. (3–5)
w,w_α,w_β	= defined in Eqs. (19–43)
$\bar{x},\bar{y}$	= Cartesian coordinates, see Fig. 1
x	= $\bar{x}q_0/R$, nondimensional axial distance
y	= $\bar{y}q_0/R$, nondimensional circumferential distance
α	= amplitude of $F^{(1)}$, defined by Eq. (15)
β_0	= angle in radians between adjacent stiffeners
γ	= $q_0GJ/(DR)$, torsional rigidity parameter
δ	= amplitude of buckling displacement, see Eq. (3)
$\bar{\delta}$	= amplitude of a small geometrical imperfection
Δ_0,Δ_n	= determinants defined by Eqs. (39) and (35), respectively
θ	= $q_0\beta_0/(2\pi)$, flatness parameter
θ_{cr}	= value of $\theta(p,\gamma)$ when $b=0$
Λ	= load parameter defined in Eq. (3)
λ	= ratio of the wavelength in the y direction to the wavelength in the x direction
ν	= Poisson's ratio
$\bar{\sigma}$	= applied axial compressive stress
σ	= $\bar{\sigma}Rc/(Et)$, nondimensional axial compressive stress
σ_{cr}	= minimum eigenvalue as a function of λ or critical buckling load
σ_s	= critical buckling load for an imperfect structure defined in Fig. 2

Received January 18, 1971; revision received May 17, 1971. This work was supported in part by NASA under Grant NGL 22-007-012, and by the Division of Engineering and Applied Physics, Harvard University.

Index Category: Structural Stability Analysis.

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1. Introduction

IN Ref. 1, the buckling and initial postbuckling behavior of an axially compressed narrow cylindrical panel is presented. Such panels occur in longitudinally stiffened